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Two Phase Communication Protocol of a Two Qubit States using Concatenated GHZ States

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Abstract: In this paper we introduce a new communication (teleportation) protocol for transferring arbitrary two qubits by using concatenated three particle entangled states as quantum channels which are robust in noisy environments. Due to almost inevitable existence of noise, which can create devastation in the communication systems, such robust quantum channels become necessary. The protocol is a perfect teleportation protocol. An advantage of the process is that only one fourth of the measurement basis elements of the sender appear in the protocol.

Keywords: Concatenation; Concatenated GHZ States; Teleportation; Bell State

1.Introduction:

Teleportation is a well known quantum communication protocol. The most important component of teleportation is the quantum teleporting channel which is essentially an entangled quantum resource shared between the distant parties. Although the original proposal of teleportation used maximally entangled Bell states as the teleportation channel [1], later it was discovered that different types of entangled resources can be used in teleportation protocols. Not only arbitrary qubits, but also other categories of quantum states can be transferred through the process of teleportation by use of appropriate quantum channels. Apart from the perfect teleportation protocols, that is, the protocols which transport the quantum state exactly, that is, with both unit probability and fidelity, there are two other types of protocols. One is imperfect protocol in which the state is transferred with fidelity less then one and the other is the probabilistic protocol in which the state is transferred with fidelity one but with a nonvanishing probability of the case of failure in which the state is irrevocably lost. Generally, but not always, the imperfect and probabilistic protocols use less entangled states whereas the perfect teleportations normally utilize maximally entangled states. There are exceptions to this generally observed fact as, for instance, arbitrary single-qubit states were perfectly teleported using class W-states which are not maximally entangled [2]. There is a large literature on various kinds of teleportation protocols which utilize pure entangled channels. Some instances of these works are in [3–10]. Particularly GHZ and GHZ-like states appear as quantum resources in a large number of works on teleportation, some of which are noted in [6,8–10]. Pure quantum channel is an idealization. There is almost inevitable existence of noise which distorts the pure quantum state. Particularly there can be interactions between the quantum channel and the environment in which case the channel becomes an open quantum system. The environment may cause two types of damping, namely, the Amplitude damping and the Phase damping which are responsible for the dissipation of energy and decoherence respectively. The noisy quantum channels for teleportation have appeared in works like those noted in [11–13]. There can be another approach to the problem of noise. Such channels can be used which are robust in a noisy environment. This is with this motivation that we introduce the use of concatenated quantum channel. Particularly we show that it is possible to teleport an arbitrary qubit using concatenated GHZ states as quantum channels. In the above we have already mentioned the importance of GHZ states in teleportation. In fact the role of GHZ states is also significant in areas like quantum metrology [14,15], distributed quantum networks [16,17], etc. It is a multipartite quantum state, which, for the case of m number of parties, is given by

$$|GHZ_m^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes m} \pm |1\rangle^{\otimes m}).$$

In a recent work by *Frӧwis et al.* [18] concatenation of GHZ states was introduced, which is

$$|\phi\rangle_{s_{m,N}} = \frac{1}{\sqrt{2}}(|GHZ_m^{+}\rangle^{\otimes N} + |GHZ_m^{-}\rangle^{\otimes N}), \qquad ...(1).$$

It was established therein that such concatenated states share the advantages of the GHZ states, but their robustness in the presence of noise is higher than the ordinary GHZ states [18]. An experimental realization of these states has been reported in [19]. We use three particle concatenated GHZ states as our quantum channel which is obtained by putting $m = 3$ and $N = 2$ in the formula (1). Explicitly, our teleporting channel is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right) \qquad ...(2).$$

One feature of the above state is that it is maximally entangled. The protocols which lead the non-maximally entangled states to states with maximal entanglement, commonly known as Entanglement Concentration Protocols (ECPs), are considered to be important because of the overwhelmingly superior usages of maximally entangled states over the states with less entanglement. Some instance of ECPs are in works [20,21]. An ECP for concatenated GHZ states has been given by Qu et al. recently in [22].

The works [19] and [22] describe the ways and means for obtaining the quantum resources given in (2).

The above is the background for our consideration of the teleportation protocol with the channel as given in (2).

2. Main Result :

An arbitrary two qubit state is defined as $|\phi\rangle_{ab} = \alpha|00\rangle_{ab} + \beta|11\rangle_{ab}$, where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. Alice wants to teleport the state $|\phi\rangle$ to Bob. To do this Alice first sends it to a third party say, Charlie and then Charlie sends it to Bob. To perform this task Alice and Charlie share a quantum channel which is given by a maximally entangled concatenated GHZ state described as

$$|\psi\rangle_{1234} = \frac{1}{\sqrt{2}}\left(\frac{|01\rangle+|10\rangle}{\sqrt{2}}\right)_{12}\left(\frac{|01\rangle+|10\rangle}{\sqrt{2}}\right)_{34} + \frac{1}{\sqrt{2}}\left(\frac{|01\rangle-|10\rangle}{\sqrt{2}}\right)_{12}\left(\frac{|01\rangle-|10\rangle}{\sqrt{2}}\right)_{34}.$$

The qubits a, b, 1 and 2 are in the possession of Alice and the qubits 3 and 4 are in the possession of Charlie. We consider the Bell states by $|\phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$ and $|\psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$ in the following. Using Bell states the composite system of the two qubit and the channel can be written as

$$\begin{aligned}
|\Psi\rangle &= |\phi\rangle_{ab} \otimes |\psi\rangle_{1234} \\
&= (\alpha|00\rangle_{ab} + \beta|11\rangle_{ab}) \otimes \frac{1}{2\sqrt{2}}\{(|01\rangle + |10\rangle)(|01\rangle + |10\rangle) + (|01\rangle - |10\rangle)(|01\rangle - |10\rangle)\}_{1234} \\
&= \frac{1}{2\sqrt{2}}[|\phi^+\rangle_{a1}|\psi^+\rangle_{b2}(\alpha|01\rangle_{34} + \beta|10\rangle_{34}) + |\phi^+\rangle_{a1}|\psi^-\rangle_{b2}(\alpha|01\rangle_{34} - \beta|10\rangle_{34}) \\
&\quad + |\phi^-\rangle_{a1}|\psi^+\rangle_{b2}(\alpha|01\rangle_{34} - \beta|10\rangle_{34}) + |\phi^-\rangle_{a1}|\psi^-\rangle_{b2}(\alpha|01\rangle_{34} + \beta|10\rangle_{34}) \\
&\quad + |\psi^+\rangle_{a1}|\psi^+\rangle_{b2}(\alpha|01\rangle_{34} + \beta|10\rangle_{34}) + |\psi^+\rangle_{a1}|\psi^-\rangle_{b2}(\alpha|01\rangle_{34} - \beta|10\rangle_{34}) \\
&\quad + |\psi^-\rangle_{a1}|\psi^+\rangle_{b2}(\alpha|01\rangle_{34} - \beta|10\rangle_{34}) + |\psi^-\rangle_{a1}|\psi^-\rangle_{b2}(\alpha|01\rangle_{34} + \beta|10\rangle_{34})] \qquad ...(3).
\end{aligned}$$

Alice performs a measurement on her qubit pairs (a, 1), and (b, 2) under Bell basis $|\phi^{\pm}\rangle$ and $|\psi^{\pm}\rangle$. After performing the measurements, Alice sends her measurement result classically to Charlie. Then Charlie applies an appropriate unitary operator to recover the unknown two-qubit state. The details are given in Table 1:.

Table 1:

Alice's Measurement	Charlie's State	Charlie's Unitary Operation
$\|\phi^+\rangle_{a1}\|\psi^+\rangle_{b2}$	$\alpha\|01\rangle_{34} + \beta\|10\rangle_{34}$	$I_3 \otimes I_4$
$\|\phi^+\rangle_{a1}\|\psi^-\rangle_{b2}$	$\alpha\|01\rangle_{34} - \beta\|10\rangle_{34}$	$(\sigma_z)_3 \otimes (\sigma_z)_4$
$\|\phi^-\rangle_{a1}\|\psi^+\rangle_{b2}$	$\alpha\|01\rangle_{34} - \beta\|10\rangle_{34}$	$(\sigma_z)_3 \otimes (\sigma_z)_4$
$\|\phi^-\rangle_{a1}\|\psi^-\rangle_{b2}$	$\alpha\|01\rangle_{34} + \beta\|10\rangle_{34}$	$I_3 \otimes I_4$
$\|\psi^+\rangle_{a1}\|\psi^+\rangle_{b2}$	$\alpha\|01\rangle_{34} + \beta\|10\rangle_{34}$	$I_3 \otimes I_4$
$\|\psi^+\rangle_{a1}\|\psi^-\rangle_{b2}$	$\alpha\|01\rangle_{34} - \beta\|10\rangle_{34}$	$(\sigma_z)_3 \otimes (\sigma_z)_4$
$\|\psi^-\rangle_{a1}\|\psi^+\rangle_{b2}$	$\alpha\|01\rangle_{34} - \beta\|10\rangle_{34}$	$(\sigma_z)_3 \otimes (\sigma_z)_4$
$\|\psi^-\rangle_{a1}\|\psi^-\rangle_{b2}$	$\alpha\|01\rangle_{34} + \beta\|10\rangle_{34}$	$I_3 \otimes I_4$

In the second phase ....Charlie sends. Charlie wants to teleport the state $|\psi\rangle_{34} = \alpha|01\rangle_{34} + \beta|10\rangle_{34}$ to Bob. To perform this task Charlie and Bob share a quantum channel which is given by a maximally entangled concatenated GHZ state described as

$$|\psi\rangle_{5678} = \frac{1}{\sqrt{2}}\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right)_{56}\left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right)_{78} + \frac{1}{\sqrt{2}}\left(\frac{|00\rangle-|11\rangle}{\sqrt{2}}\right)_{56}\left(\frac{|00\rangle-|11\rangle}{\sqrt{2}}\right)_{78}.$$

The qubits 3, 4, 5 and 6 are in the possession of Charlie and the qubits 7 and 8 are in the possession of Bob. We consider the Bell states by

$$|\phi^{\pm}\rangle = \frac{|00\rangle\pm|11\rangle}{\sqrt{2}} \text{ and } |\psi^{\pm}\rangle = \frac{|01\rangle\pm|10\rangle}{\sqrt{2}}$$

in the following. Using Bell states the composite system of the two qubit and the channel can be written as

$$\begin{aligned}
|\Psi\rangle &= |\psi\rangle_{34} \otimes |\psi\rangle_{5678} \\
&= (\alpha|01\rangle_{34} + \beta|10\rangle_{34}) \otimes \frac{1}{2\sqrt{2}}\{(|00\rangle + |11\rangle)(|00\rangle + |11\rangle) + (|00\rangle - |11\rangle)(|00\rangle - |11\rangle)\}_{5678} \\
&= \frac{1}{2\sqrt{2}}[|\phi^+\rangle_{35}|\psi^+\rangle_{46}(\alpha|00\rangle_{78} + \beta|11\rangle_{78}) + |\phi^+\rangle_{35}|\psi^-\rangle_{46}(-\alpha|00\rangle_{78} + \beta|11\rangle_{78}) \\
&\quad + |\phi^-\rangle_{35}|\psi^+\rangle_{46}(\alpha|00\rangle_{78} - \beta|11\rangle_{78}) + |\phi^-\rangle_{35}|\psi^-\rangle_{46}(-\alpha|00\rangle_{78} - \beta|11\rangle_{78}) \\
&\quad + |\psi^+\rangle_{35}|\phi^+\rangle_{46}(\beta|00\rangle_{78} + \alpha|11\rangle_{78}) + |\psi^+\rangle_{35}|\phi^-\rangle_{46}(\beta|00\rangle_{78} - \alpha|11\rangle_{78}) \\
&\quad + |\psi^-\rangle_{35}|\phi^+\rangle_{46}(\beta|00\rangle_{78} + \alpha|11\rangle_{78}) + |\psi^-\rangle_{35}|\phi^-\rangle_{46}(\beta|00\rangle_{78} - \alpha|11\rangle_{78})] \qquad ...(4).
\end{aligned}$$

Charlie performs a measurement on her qubit pairs (3, 5),and (4, 6) under Bell basis $|\phi^{\pm}\rangle$ and $|\psi^{\pm}\rangle$. After performing the measurements, Charlie sends her measurement result classically to Bob. Then Bob applies an appropriate unitary operator to recover the unknown two-qubit state. The details are given in Table 1:.

Table 2:

Charlie's Measurement	Bob's State	Bob's Unitary Operation
$\lvert\phi^+\rangle_{35}\lvert\psi^+\rangle_{46}$	$\alpha\lvert 00\rangle_{78} + \beta\lvert 11\rangle_{78}$	$I_7 \otimes I_8$
$\lvert\phi^+\rangle_{35}\lvert\psi^-\rangle_{46}$	$-\alpha\lvert 00\rangle_{78} + \beta\lvert 11\rangle_{78}$	$-I_7 \otimes (\sigma_z)_8$
$\lvert\phi^-\rangle_{35}\lvert\psi^+\rangle_{46}$	$\alpha\lvert 00\rangle_{78} - \beta\lvert 11\rangle_{78}$	$(\sigma_z)_7 \otimes I_8$
$\lvert\phi^-\rangle_{a1}\lvert\psi^-\rangle_{b2}$	$-\alpha\lvert 01\rangle_{34} - \beta\lvert 10\rangle_{34}$	$-I_7 \otimes I_8$
$\lvert\psi^+\rangle_{35}\lvert\phi^+\rangle_{46}$	$\beta\lvert 00\rangle_{78} + \alpha\lvert 11\rangle_{78}$	$(\sigma_x)_7 \otimes (\sigma_x)_8$
$\lvert\psi^+\rangle_{35}\lvert\phi^-\rangle_{46}$	$\beta\lvert 00\rangle_{78} - \alpha\lvert 11\rangle_{78}$	$(-\sigma_x)_7 \otimes (i\sigma_y)_8$
$\lvert\psi^-\rangle_{35}\lvert\phi^+\rangle_{46}$	$\beta\lvert 00\rangle_{78} + \alpha\lvert 11\rangle_{78}$	$(\sigma_x)_7 \otimes (\sigma_x)_8$
$\lvert\psi^-\rangle_{35}\lvert\phi^-\rangle_{46}$	$\beta\lvert 00\rangle_{78} - \alpha\lvert 11\rangle_{78}$	$(-\sigma_x)_7 \otimes (i\sigma_y)_8$

From the expression (3) and (4) both from the Table 1 and Table 2 that out of 16 basis elements in the measurement basis, only eight appear in the Alice's measurement and Charlie's measurement. This leads to substantial advantage of the protocol.

3. Discussion and Conclusion:

A primary concern in all types of communications, both classical and quantum, is the existence of noise. Quantum noise makes the channel into an open quantum system whose dynamics can frustrate the goal of the communication. It is necessary to use resources of communications which are not substantially affected by noise. The problem is supposed to open a vast field of study. The introduction of concatenation in communication process, originally studied for coding purpose [23–25], is a step towards the goal of research mentioned above. The importance of the present work lies in it.

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